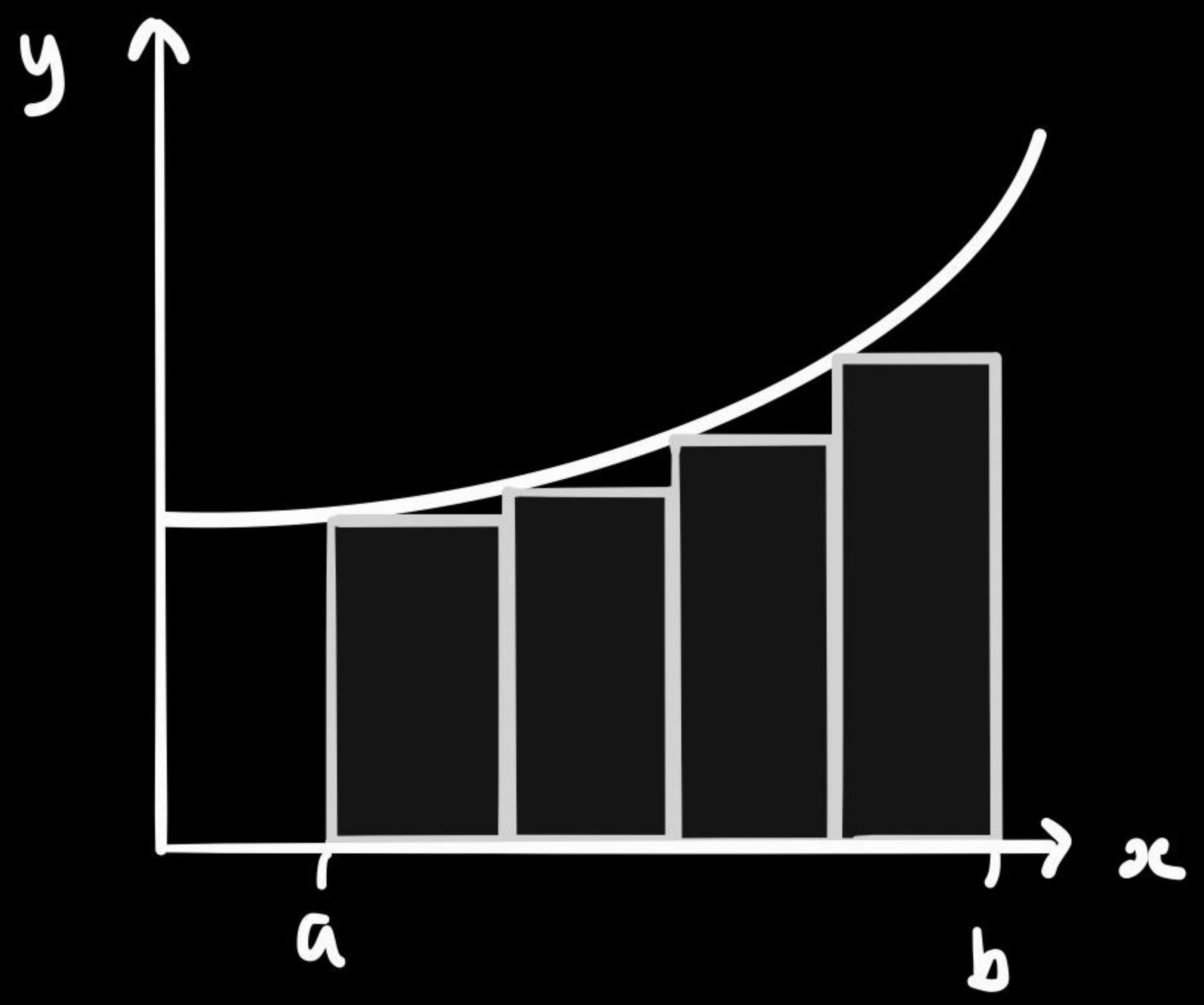
Riemann Integrals

By Maya

Riemann Sums

We can calculate the approximate area under a curve by partitioning the region into shapes.

Here, we have taken the left Riemann sum which uses the minimum value of the function within each interval which leaves us with an underestimation as this function is monotonically increasing - it is always increasing or constant. If the function was monotonically decreasing the left Riemann sum would be an overestimation.

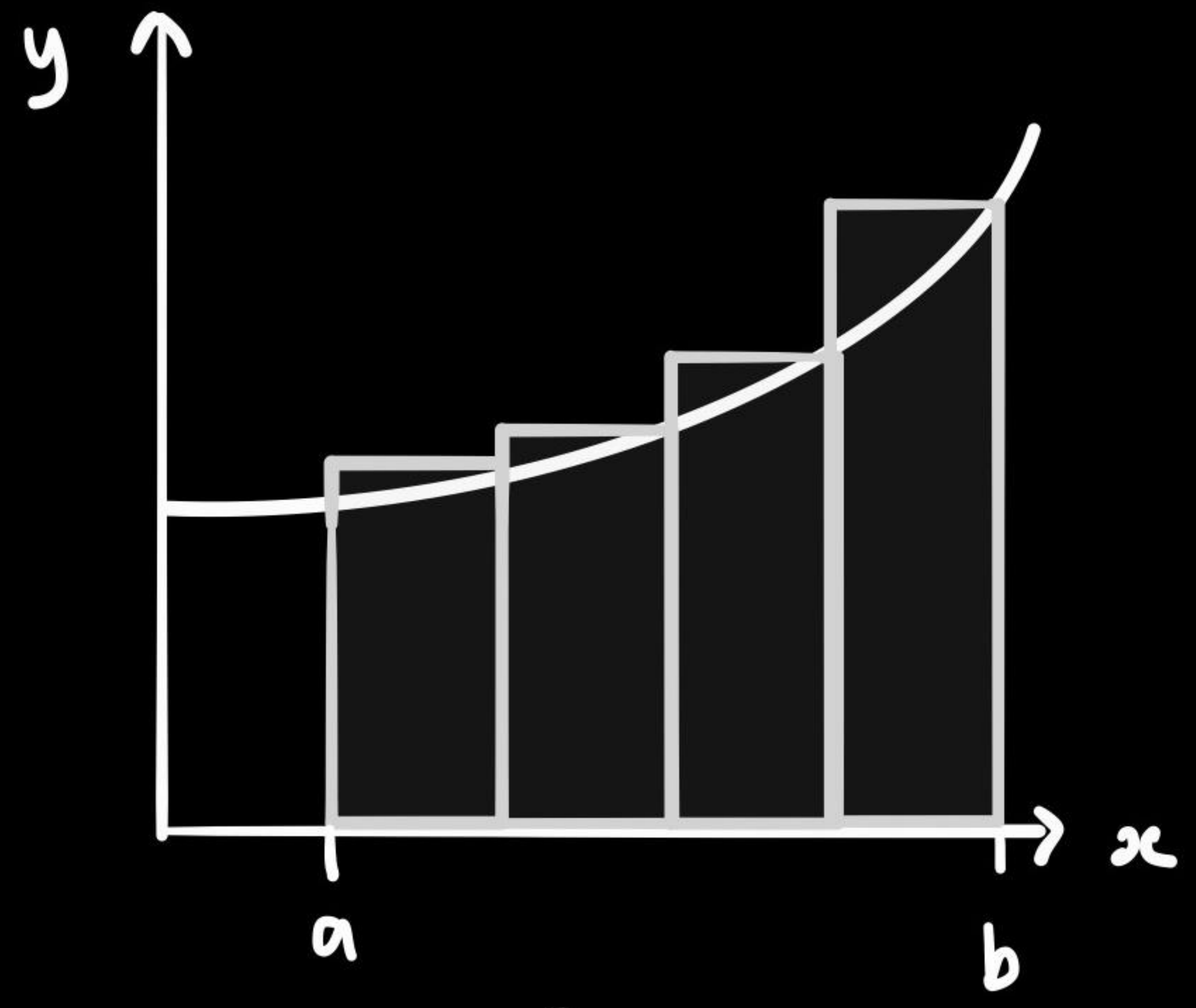


$$A \approx \sum_{r=1}^{n} f(x_{r-1}) \Delta x$$

$$\Delta x = \frac{b-a}{n}$$

Riemann Sums

Similarly, here we have taken the right Riemann sum which uses the maximum value of the function within each interval which leaves us with an overestimation as this function is monotonically increasing. With a monotonically decreasing function this would be an underestimation.



$$A \approx \sum_{r=1}^{n} f(x_{r-1})\,\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

The Trapezium Rule

The trapezium rule is a type of Riemann sum. It is equal to the average of the left and right Riemann sums for the same function.

$$A = \frac{1}{2} h \left(b_1 + b_2 \right)$$

$$I = \int_a^b y \, dx \approx \frac{1}{2} h \left\{ (y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1}) \right\}$$

$$h = \frac{b-a}{n}$$

Riemann Integrals

A Riemann integral is the limit of a Riemann sum as n tends to infinity. Essentially we have more and more rectangles and this leads to a better approximation of the area under the curve. As n tends to infinity both the under and over estimates will be equal.

$$I = \int_a^b f(x)\,dx$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

Where $\Delta x = \frac{b-a}{n}$

Riemann Integrals and Riemann Sums

Question 1.

Which integral is equal to this Riemann sum?

$$\lim_{n\to\infty} \sum_{i=1}^{n} \ln\left(2 + \frac{5i}{n}\right) \cdot \frac{5}{n}$$

A $\int_2^7 \ln x \, dx$

B $\int_0^5 \ln x \, dx$

C $\int_2^5 \ln x \, dx$

D $\int_0^7 \ln x \, dx$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + \Delta x i)\,\Delta x$$

Where $\Delta x = \frac{b-a}{n}$

We need to find the limits by comparing the top right to the sum

$\Delta x = \frac{b-a}{n} = \frac{5}{n}$

$b - a = 5$ (1)

$a = 2$ (2)

(2) in (1)

$b - 2 = 5$

$b = 7$

$\therefore$ A is the answer

Riemann Integrals and Riemann Sums

Question 2.

Which Riemann Sum is equal to this integral?

$$\int_0^{\pi} \sin x \, dx$$

A $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi i}{n}\right) \cdot \frac{\pi}{n}$

B $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi i}{n}\right) \cdot \frac{i}{n}$

C $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i}{n}\right) \cdot \frac{i}{n}$

D $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i}{n}\right) \cdot \frac{\pi}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + \Delta x i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$

We need to find Δx

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{n} = \frac{\pi}{n}$$

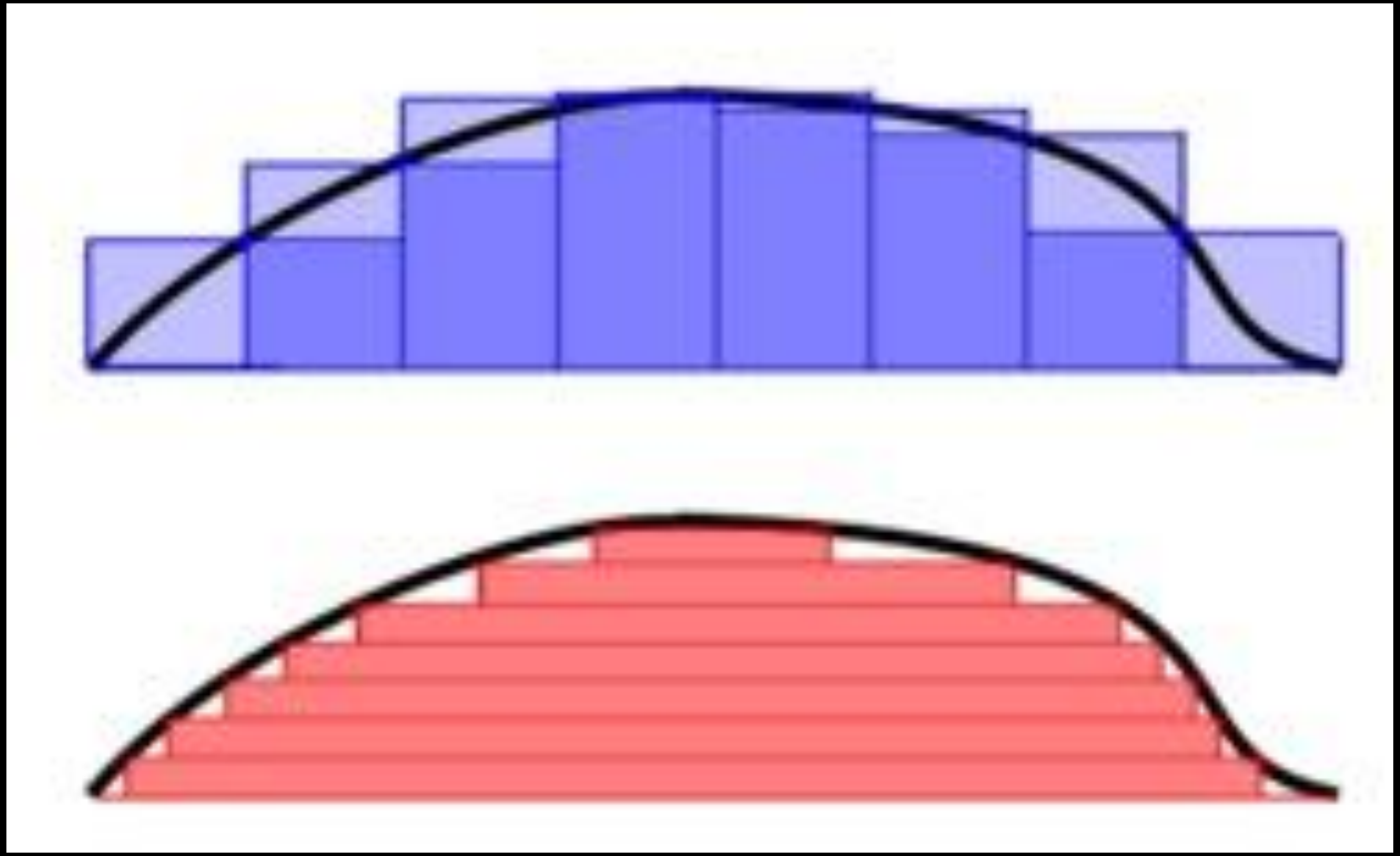
by using the top right the right answer is A

Problems With Riemann Integrals

There are many integrals that “should be” integrable, but that are not Riemann integrable. An integral of the form

$$I = \int_a^b f(x)\,dx$$

is only Riemann integrable if it is continuous on the interval [a,b]. There are many other definitions of integrals such as the Lebesgue integral and the Darboux integral that are more helpful in certain situations.



Riemann Integral (top) compared with the Lebesgue Integral (bottom). The Lebesgue integral takes horizontal slabs as opposed to vertical rectangles.